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The semigroups of order 9 and their automorphism groups

Andreas Distler · Tom Kelsey

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Abstract We report the number of semigroups with 9 elements up to isomorphism or anti-isomorphism to be 52 989 400 714 478 and up to isomorphism to be 105 978 177 936 292. We obtained these results by combining computer search with recently published formulae for the number of nilpotent semigroups of degree 3. We further provide a complete account of the automorphism groups of the semigroups with at most 9 elements. We use this information to deduce that there are 148 195 347 518 186 distinct associative binary operations on an 8-element set and 38 447 365 355 811 944 462 on a 9-element set.

Keywords Semigroup · Automorphism group · Enumeration

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1 Introduction

Classification of finite semigroups of a given order goes back to the 1950s when Tamura undertook hand calculations, first for orders 2 and 3 [22] and later for order 4 [23]. Around the same time Forsythe introduced computer search to the problem [8] implementing a backtrack algorithm to find semigroups on a 4 element set. Subsequently various authors refined his approach [18, 20, 15, 21], so that by 1994 semigroups were classified up to order 8. These semigroups are nowadays available in the data library Smallsemi [6]. A recent

Andreas Distler
Centro de Álgebra da Universidade de Lisboa, 1649-003 Lisboa, Portugal
Tel.: +351 21 790 4716
Fax: +351 21 795 4288
E-mail: adistler@fc.ul.pt

Tom Kelsey
School of Computer Science, University of St Andrews, KY16 9SX, UK

advance in the enumeration of semigroups are the formulae derived in [7] for the numbers of semigroups S for which $|\{abc \mid a,b,c \in S\}| = 1$ and S is not a zero semigroup. Such semigroups are called *nilpotent of degree 3*. These were used in the attempt to establish an asymptotic lower bound on the number of all semigroups on a finite set [17]. The analysis in [21] shows that for order 8 about 99.4% of the semigroups are nilpotent of degree 3.

Research investigating the automorphism groups of semigroups of a given order has a far briefer history. It has long been known that every group appears as the automorphism group of some semigroup which is a consequence of the analogous result for graphs [10], but the first algorithm to compute automorphism groups of semigroups was only presented in [1] where Araújo *et al.* compute as one application of their general method the automorphism groups of semigroups of orders at most 7.

Naturally the principal aim of the aforementioned investigations was to consider 'structural types' of semigroups of the given order rather than distinct semigroups on a set of that size. Two semigroups S and R are *anti-isomorphic* if one is isomorphic to the dual of the other, that is if there exists a bijection $\sigma : S \to R$ such that $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in S$; in this case σ is an *anti-isomorphism*. For short we write *(anti-)isomorphic* to mean isomorphic or anti-isomorphic and analogously write *(anti-)isomorphism*. Classification of semigroups has mainly been done up to (anti-)isomorphism. The connection to a classification up to isomorphism is provided by those semigroups which are anti-isomorphic to themselves, that is isomorphic to their dual, called *self-dual*.

In this paper we enumerate semigroups of order 9 up to isomorphism and (anti-)isomorphism. The number up to (anti-)isomorphism, 52 989 400 714 478, was first reported in [5], without the explanation and justification that are provided here. The number up to isomorphism is 105 978 177 936 292. We also classify the semigroups of orders 8 and 9 by their automorphism groups (see Tables 10 and 11) and deduce that the number of distinct semigroups on a set with 8 elements is 148 195 347 518 186 and on a set with 9 elements is 38 447 365 355 811 944 462. We find that only a small proportion of the subgroups of the symmetric groups of degrees 8 and 9 are isomorphic to the automorphism group of any such semigroup (Table 13) and prove in particular that the automorphism group of a semigroup is transitive if and only if it is a rectangular band (Propostion 1).

We obtain semigroups and their automorphism groups by computer search. For the enumeration it suffices to count those semigroup that are *not* nilpotent of degree 3, since their numbers are known [7]. We perform a series of computations to find multiplication tables of the remaining semigroups, utilising an approach similar to that described in [5]: we model the search as a family of constraint satisfaction problems and execute the constraint solver Minion [12] to get the solutions. The computer algebra system GAP [11] is used in the preparation of the input files, in particular for calculations to avoid (anti-)isomorphic solutions. We also use Minion to search for automorphism groups and GAP to identify their isomorphism types.

In the forthcoming section we explain how we represent semigroups and isomorphisms respectively anti-isomorphism between them in an adequate way for the computer search. Section 3 contains an introduction to Constraint Satisfaction, and a description of a formal model for finding canonical representatives of semigroups of a given order up to (anti-)isomorphism. We also describe adaptations of the model that allow us to find automorphism groups of semigroups, and to find self-dual semigroups. In Section 4 we replace the initial model with an equivalent family of models following an idea introduced by Plemmons in the classification of semigroups of order 6 [20]. Additionally we extend the idea proving a unified framework for this approach. The extended approach allows one to incorporate mathematical knowledge more easily into the search for particular types of semigroups, a method that we apply in Section 5 to the enumeration of bands. In Section 6 we report our computational experience and give detailed classification results for semigroups of order 9 by various properties. For comparative purposes, we also show the equivalent results for smaller orders in certain cases. In the final section we describe our approach to the computation of the automorphism groups of semigroups. We present results for semigroups of order at most 9 up to isomorphism and up to (anti-)isomorphism. We use this knowledge to calculate the numbers of distinct semigroups on a set with at most 9 elements, and discuss which subgroups of the symmetric group are isomorphic to the automorphism group of a semigroup.

2 Preliminaries

In this paper the elements of a semigroup will mostly be $\{1, 2, \ldots, n\}, n \in \mathbb{N}$ which we abbreviate by $[n]$. As usual the multiplication table of a semigroup S is the square matrix $T_S = (t_{a,b})_{a,b \in S}$ with $t_{a,b} = ab$. If the underlying set of S is $[n]$ we may assume that the rows and columns of the multiplication table are indexed according to their position in the table which allows us to omit the row and column header. Under this convention every square matrix of size n with entries in $[n]$ uniquely defines a binary operation on $[n]$ and we denote the set of all such matrices by Ω_n.

We want to describe isomorphisms and anti-isomorphisms in terms of multiplication tables. Consider first two isomorphic semigroups S and R on $[n]$ and a permutation π in the symmetric group S_n which is an isomorphism from S to R. Given that $T_S = (t_{i,j})_{i,j \in [n]}$ is the multiplication table of S it follows that $((t_{i,j})^\pi)_{i^\pi, j^\pi \in [n]}$ is the multiplication table of R. And if S and R are anti-isomorphic and π an anti-isomorphism from S to R then $((t_{j,i})^\pi)_{i^\pi, j^\pi \in [n]}$ is the multiplication table of R. Hence we can capture isomorphism and anti-isomorphism in the following action $\phi : \Omega_n \times (S_n \times C_2) \to \Omega_n$ sending a multiplication table $T \in \Omega_n$ to

$$T^{(\pi,c)} = \begin{cases} ((t_{i,j})^\pi)_{i^\pi, j^\pi \in [n]} & \text{if } c = 1_{C_2} \\ ((t_{j,i})^\pi)_{i^\pi, j^\pi \in [n]} & \text{otherwise.} \end{cases} \tag{1}$$

The orbits of this action are sets of those multiplication tables which define (anti-)isomorphic binary operations.

To avoid confusion and repetition of similar arguments we will throughout the paper use the action given in (1) only, incorporating both isomorphism and anti-isomorphism. Considerations up to isomorphism using an action of S_n on Ω_n are left to the reader.

3 Enumeration using Constraint Satisfaction

As in previous classifications of semigroups we search their multiplication tables. To formalise our task we use the language of Constraint Satisfaction, a technique developed to model and solve discrete combinatorial problems, and start by giving basic definitions.

Definition 1 A *constraint satisfaction problem (CSP)* is a triple (V, D, C), consisting of a finite set V of *variables*, a finite set D, called the *domain*, of *values*, and a set C containing subsets of $\{h \mid h : V \to D\}$ called *constraints*.

In practice, instead of being subsets of the set of all functions from V to D, constraints are formulated as conditions defining such subsets. It then becomes intuitively clear that one is looking for assignments of values in the domain of a CSP to all variables such that no constraint is violated. This idea is formalised in the next definition.

Definition 2 Let $L = (V, D, C)$ be a CSP. A partial function $p : V \to D$ is an *instantiation*. An instantiation p *satisfies* a constraint if there exists a function h in the constraint, such that $h(x) = p(x)$ for all $x \in V$ on which p is defined. An instantiation is *valid*, if it satisfies all the constraints in C. An instantiation defined on all variables is *total*. A valid, total instantiation is a *solution* to L. The number of solutions of L will be denoted by $\#L$.

3.1 Counting all semigroups

We formulate a CSP which has those multiplication tables in Ω_n as solutions that define an associative multiplication:

CSP 1 *For $n \in \mathbb{N}$ define a CSP $L_n = (V_n, D_n, C_n)$. The set V_n consists of n^2 variables $\{t_{i,j} \mid 1 \leq i, j \leq n\}$, one for each position in an $(n \times n)$-multiplication table, having domain $D_n = [n]$. The constraints in C_n are*

$$t_{t_{i,j},k} = t_{i,t_{j,k}} \text{ for all } i, j, k \in [n], \tag{2}$$

reflecting associativity. (Note that (2) is a slight abuse of notation: using a variable as an index shall refer to its value.)

The multiplication table defined by a solution of L_n from CSP 1 will be associative due to the constraints C_n and, in turn, the table of every associative multiplication fulfils the constraints in C_n. Thus the valid, total instantiations for L_n correspond to the semigroups on $[n]$. As the constraints C_n enforcing associativity will be present in every following model, the solutions will always define semigroups and are often referred to as such.

The number of all different semigroups on $[n]$ grows rapidly with n and most of the semigroups are nilpotent of degree 3 [17]. As the construction for nilpotent semigroups of degree 3 on $[n]$ is also given in [17], they do not have to be searched for. We forbid these by requiring that not all multiplications of three elements give the same result. Adding the constraint

$$\exists i, j, k, q, r, s \in [n] : t_{i,t_{j,k}} \neq t_{q,t_{r,s}} \tag{3}$$

to C_n yields the CSP, denoted as L_n^{-3}, having as solutions all different semigroups on $[n]$, which are neither nilpotent of degree 3 nor a zero semigroup.

3.2 Counting up to (anti-)isomorphism

Our primary aim is not to find all semigroups on $[n]$, but rather to find representatives for all types of structurally different semigroups, where structurally different means up to (anti-)isomorphism.

With increasing n it becomes – due to the large number of solutions – very quickly impractical to test for every two semigroups from the solutions of L_n or L_n^{-3} whether they are (anti-)isomorphic. Instead we shall define a canonical solution for each class following a standard approach in the classification of algebraic and combinatorial structures. To make the test for canonicity an integral part of the CSP we adapt a common symmetry breaking technique from Constraint Satisfaction.

We first need another way to describe a solution of a CSP. A *literal* (also called *variable-value pair*) of a CSP $L = (V, D, C)$ is an element in the Cartesian product $V \times D$. Literals are denoted in the form $(x = k)$ with $x \in V$ and $k \in D$. An instantiation p corresponds to the set of literals $\{(x = p(x)) \mid p \text{ is defined on } x\}$, which uniquely determines p (but not every set of literals yields an instantiation). In particular we get an action of $S_n \times C_2$ on literals, induced from the action (1) on multiplication tables.

$$(t_{i,j} = k)^{(\pi,c)} = \begin{cases} (t_{i^\pi,j^\pi} = k^\pi) & \text{if } c = 1_{C_2} \\ (t_{j^\pi,i^\pi} = k^\pi) & \text{otherwise.} \end{cases} \tag{4}$$

Given a fixed ordering $(\chi_1, \chi_2, \ldots, \chi_{|V||D|})$ of all literals in $V \times D$, an instantiation p can be represented as a bit vector of length $|V||D|$. The bit in the i-th position is 1 if χ_i is contained in the set of literals corresponding to p and otherwise the bit is 0. The resulting bit vector for the instantiation p will be denoted by $(\chi_1, \chi_2, \ldots, \chi_{|V||D|})_p$.

Sets of solutions of L_n and L_n^{-3} that lead to (anti-)isomorphic semigroups are orbits under the action of $S_n \times C_2$ given in (1). There is one solution in each orbit for which the corresponding bit vector is lexicographic maximal, which we take to be the property identifying the canonical solution in the orbit. We denote the standard lexicographic order on vectors by $\prec$. We obtain a new CSP $\overline{L}_n$ from L_n, respectively $\overline{L}_n^{-3}$ from L_n^{-3}, by adding, for all non-identity $g \in S_n \times C_2$, the constraint consisting of those functions $h : V \to D$ for which

$$(\chi_1^g, \chi_2^g, \ldots, \chi_{|V||D|}^g)_h \preceq (\chi_1, \chi_2, \ldots, \chi_{|V||D|})_h. \tag{5}$$

The solutions of the new CSPs are pairwise not (anti-)isomorphic, because $\overline{L}_n$ respectively $\overline{L}_n^{-3}$ have as solutions all canonical tables from orbits of solutions of L_n respectively L_n^{-3}. Hence $\#\overline{L}_n$ equals the number of semigroups of order n up to (anti-)isomorphism, while $\#\overline{L}_n^{-3}$ equals the number of semigroups of order n that are not nilpotent of degree 3 nor a zero semigroup up to (anti-)isomorphism.

There is a computational drawback of the method explained in this section to avoid (anti-)isomorphic solutions: the number of canonicity constraints (5) to be added to L_n to obtain $\overline{L}_n$ is $2n! - 1$ and their length is n^3, which makes the space requirements for the formulation of the constraints grow very large already for small values of n. We can improve the situation to some extent by shortening in (5) the vectors on both sides depending on g without influencing the constraint. The technique we use is based on [9, Rule 1] and the easiest example of its application is the removal of literals which appear at the same position in both vectors. It remains the more significant problem that the number of constraints grows superexponentially with n. In Section 4 we will explain an approach that ultimately overcomes this obstacle in our specific enumeration problem.

3.3 Finding automorphisms and self-dual semigroups

A straightforward variation of the canonicity constraints (5) allows to identify or prescribe automorphisms of solutions of the CSP. A bijection $\pi \in S_n$ is an automorphism if equality holds in constraint (5) corresponding to $(\pi, 1_{C_2})$. We can now either record for each solution for which of the constraints equality holds; or alternatively specify the automorphism group in advance, requiring equality for the constraints corresponding to permutations in the chosen group and strict inequality for all other permutations.

A similar approach can be used to identify self-dual semigroups. If we require equality to hold for at least one of the constraints (5) corresponding to an anti-isomorphism (that is an element in $S_n \times C_2$ with non-trivial C_2 component) then the solutions will be exactly the self-dual semigroups of the original CSP.

4 Families of CSPs

Possible enhancements of the CSP L_n are restricted by the fact that not much can be said about the multiplication table of a semigroup in general without knowing any of the entries. We adapt an idea from [20] to make the search for multiplication tables of semigroups more efficient. Instead of running a single computation, the search is split into cases depending on the diagonal of the multiplication table. A major advantage of this approach is that not all diagonals have to be considered when searching for semigroups up to (anti-)isomorphism.

In our adaptation we formulate one CSP for every diagonal. Note that the diagonals of multiplication tables naturally correspond to functions from $[n]$ to itself. For a table $T = (t_{i,j})_{i,j\in[n]}$ define a function $f_T : [n] \to [n], i \mapsto t_{i,i}$. Two tables can lead to the same function, but the correspondence between diagonals and functions from $[n]$ to itself is a bijection.

CSP 2 *Given a function $f : [n] \to [n]$ define a CSP $L_f = (V_n, D_n, C_f)$ based on $L_n = (V_n, D_n, C_n)$ from CSP 1 by adding for all $i \in [n]$ the constraint*

$$t_{i,i} = f(i) \tag{6}$$

to C_n to obtain C_f.

The solutions to L_f are all multiplication tables in Ω_n defining a semigroup in which the square of the element i is given by $f(i)$. In other words, the entries on the diagonal of the multiplication table are specified *a priori*. We note that for some functions f the CSP L_f will not have any solutions. For example every finite semigroup has at least one idempotent, which yields that every function without a fixed point leads to a CSP without solutions.

For a set $\mathcal{F}$ of functions from $[n]$ to $[n]$, denote by $\mathcal{L}_\mathcal{F}$ the family of CSPs $\{L_f \mid f \in \mathcal{F}\}$. Let $\mathcal{F}_n$ denote the set of all functions with at least one fixed point from $[n]$ to $[n]$. Then the CSPs in $\mathcal{L}_{\mathcal{F}_n}$ have together the same solutions as L_n. To select a smaller subset of functions in $\mathcal{F}_n$ such that the corresponding instances still contain every type of semigroup of order n up to (anti-)isomorphism, we use the following lemma which gives conditions in a general setting. We shall apply it again to a different family of CSPs in Section 5. Many more applications can be found in [4, Chapter 5].

Lemma 1 *Let $\mathcal{L} = \{L_x \mid x \in X\}$ be a family of CSPs with disjoint solution sets, and let $\mathcal{T}$ be a superset of all solutions. Further let*

$$\phi : \mathcal{T} \times G \to \mathcal{T}, (T, g) \mapsto T^g$$

be an action of a group G mapping solutions to solutions and let $\psi : \mathcal{T} \to X$ be a surjective function.

If each solution T of one of the CSPs in $\mathcal{L}$ is a solution of $L_{\psi(T)}$, and if ϕ^ψ is an induced action of G on X (that is, $x^g = \psi(T^g)$ for $x = \psi(T)$ is well-defined), then the following statements hold.

(i) *Let $Y \subseteq X$ contain at least one element of every orbit from X under the induced action ϕ^ψ. Then the solutions of $\{L_y \mid y \in Y\}$ contain at least one element from every orbit of solutions under the action of ϕ.*
(ii) *Let $S \in L_x$ and $T \in L_y$. If S is equivalent to T, then x is equivalent to y.*
(iii) *Let $T \in L_x$. Then the set of solutions of L_x equivalent to T equals the orbit of T under the stabiliser of x in G.*

Proof (i): Let T be a solution of one of the CSPs in $\mathcal{L}$. By assumption T is a solution of $L_{\psi(T)}$ and there exists a $y \in Y$ equivalent to $\psi(T)$, that is $\psi(T)^g = y$ for some $g \in G$. As $\psi(T)^g = \psi(T^g)$, it follows that T^g is a solution of $L_y = L_{\psi(T^g)}$.

(ii): Let T be equivalent to S. Thus $T^g = S$ for some $g \in G$. Note that $x = \psi(S)$ and $y = \psi(T)$ as the solution sets of different CSPs in $\mathcal{L}$ are disjoint. Hence, $x = \psi(S) = \psi(T^g) = \psi(T)^g = y^g$, showing that x is equivalent to y.

(iii): Let $g \in G$ be arbitrary. Then T^g is a solution of $L_{\psi(T^g)}$. Since the CSPs in $\mathcal{L}$ have disjoint solution sets, T^g is a solution of L_x if and only if $\psi(T^g) = x^g = x$. Hence T^g is a solution of L_x if and only if g lies in the stabiliser of x in G. □

Choosing $\mathcal{L} = \mathcal{L}_{\mathcal{F}_n}$, $\mathcal{T}$ to be Ω_n, ϕ to be the action defined in (1), and ψ as the mapping sending multiplication tables to the function corresponding to their diagonal, satisfies the conditions in Lemma 1. To obtain a set of non-equivalent functions in $\mathcal{F}_n$ under the induced action ϕ^ψ is then a reformulation of a well-known problem: the equivalence classes of functions are in one-to-one correspondence with unlabelled functional digraphs, that is directed graphs in which every vertex has outdegree 1. The construction of diagonals and the role they play in the multiplication tables of semigroups is discussed in detail in [4, Chapter 3]. If $\overline{\mathcal{F}}_n$ denotes a set of representatives of non-equivalent functions in $\mathcal{F}_n$ then each structural type of semigroup appears as solution of $\mathcal{L}_{\overline{\mathcal{F}}_n}$ due to Lemma 1(i). Moreover, different CSPs in $\mathcal{L}_{\overline{\mathcal{F}}_n}$ have pairwise not (anti-)isomorphic solutions due to the contraposition of Lemma 1(ii). This allows us to search independently in different CSPs for solutions up to (anti-)isomorphism.

The solutions of L_f form orbits under the stabiliser of f in $S_n \times C_2$ according to Lemma 1(iii). Following the considerations in Section 3.2 we add the canonicity constraint (5) for every non-identity element in the stabiliser to L_f to obtain a CSP $\overline{L}_f$ with one solution from every orbit. Hence the solutions of $\overline{\mathcal{L}}_{\overline{\mathcal{F}}_n} = \{\overline{L}_f \mid f \in \overline{\mathcal{F}}_n\}$ form a set of semigroups on $[n]$ up to (anti-)isomorphism. As before we define a CSP L_f^{-3} by adding constraint (3) to L_f, ruling out zero semigroups and nilpotent semigroups of degree 3. Adding this constraint is not necessary for all functions f, since L_f does not always allow solutions that are nilpotent of degree 3. In particular, f must not have more than one fixed point. The family of CSPs $\{L_f^{-3} \mid f \in \overline{\mathcal{F}}_n\}$ is denoted by $\mathcal{L}_{\overline{\mathcal{F}}_n}^{-3}$ and analogously we get

$$\overline{\mathcal{L}}_{\overline{\mathcal{F}}_n}^{-3} = \{\overline{L}_f^{-3} \mid f \in \overline{\mathcal{F}}_n\}. \tag{7}$$

Calculating the stabiliser in $S_n \times C_2$ of a function f corresponding to a diagonal directly under the induced action is not very efficient. This can

be avoided by reformulating the action to a pointwise action on sets. The reformulation was in principal already introduced in Section 3.2. Every element $g \in S_n \times C_2$ induces a bijection of the literals of the CSP L_f. Take the set of literals $\chi_f = \{(t_{i,i} = f(i)) \mid 1 \leq i \leq n\}$ corresponding to the given diagonal entries. Then g is in the stabiliser of f if and only if $\chi_f^g = \chi_f$. It is not a coincidence that the stabiliser of f in $S_n \times C_2$ equals the stabiliser of a set of literals, as shown by the following result complementing Lemma 1.

Lemma 2 *Let $L = (V, D, C)$ be a CSP with non-empty solution set and let $\phi : (V \times D) \times G \to V \times D$ be an action on the literals sending instantiations to instantiations. Denote the setwise stabiliser of χ in G by $\mathrm{Stab}_G(\chi)$.*

If there exists a subgroup $H \leq G$ such that each set of equivalent solutions of L is an orbit under H, and if there exists a subset of all literals $\chi \subseteq V \times D$ such that the solutions of L are the subsets of χ that are total instantiations, then each set of equivalent solutions forms an orbit under $\mathrm{Stab}_G(\chi)$.

Proof Denote the set of solutions of L by $\mathcal{T}$. For every $T \in \mathcal{T}$ and every element $g \in \mathrm{Stab}_G(\chi)$ it follows from $T^g \subseteq \chi^g = \chi$ that T^g is in $\mathcal{T}$.

It remains to show that $H \leq \mathrm{Stab}_G(\chi)$. Note that χ equals the union of all solutions. Let $h \in H$ then

$$\chi^h = \left(\bigcup_{T \in \mathcal{T}} T \right)^h = \bigcup_{T \in \mathcal{T}} T^h = \bigcup_{T \in \mathcal{T}} T = \chi$$

and hence $h \in \mathrm{Stab}_G(\chi)$. $\square$

Lemma 2 does not directly apply to the CSPs in $\mathcal{L}_{\mathcal{F}_n}$, because of the associativity constraint. If one neglects this constraint, such that the solutions are all tables fulfilling the remaining constraints, then the assumptions of Lemma 2 are satisfied. Any total instantiation for which the values on the diagonal are in χ_f is a solution for $f \in \mathcal{F}_n$, and equivalent solutions form orbits under the stabiliser of the literals in $S_n \times C_2$. Adding the associativity constraint back in does not change this fact, since associativity is invariant under isomorphism and anti-isomorphism.

Having a family of CSPs depending on the diagonal is not enough to resolve the computational bottleneck mentioned at the end of Section 3.2. The number of constraints is still $2n! - 1$ if f equals the identity function on $[n]$. In the next section we show how to avoid this problem by applying the technique from Lemma 1 again.

5 Enumeration of bands

If f is the identity function id_n on $[n]$ then the solutions of L_f as defined in CSP 2 are the bands on $[n]$. The structure of bands is well understood, knowledge that we shall use together with Lemma 1 to substitute L_{id_n} with a family of CSPs.

For the search we rely on a classification of rectangular bands. Every rectangular band is isomorphic to a semigroup on a Cartesian product $I \times \Lambda$ with multiplication defined by $(i, \lambda)(j, \mu) = (i, \mu)$, and each such multiplication defines a rectangular band. Two rectangular bands $I_1 \times \Lambda_1$ and $I_2 \times \Lambda_2$ are isomorphic if and only if $|I_1| = |I_2|$ and $|\Lambda_1| = |\Lambda_2|$, and they are anti-isomorphic if and only if $|I_1| = |\Lambda_2|$ and $|\Lambda_1| = |I_2|$. Hence, the number of rectangular bands on $[n]$ up to (anti-)isomorphism equals the number of divisors of n that are less than or equal to $\sqrt{n}$.

Every band is a semilattice of rectangular bands [3]. To define a family of CSPs we use the trivial consequence that the minimal $\mathcal{D}$-class of a band is a rectangular band.

CSP 3 *Given a rectangular band $R \subseteq [n]$ define a CSP $B_R = (V_n, D_n, C_R)$ based on $L_{\mathrm{id}_n} = (V_n, D_n, C_{\mathrm{id}_n})$ by adding the constraints*

$$t_{i,j} = ij \text{ if } i, j \in R \tag{8}$$
$$t_{i,j}, t_{j,i} \in R \text{ if } i \in R, j \in [n] \tag{9}$$

to C_{id_n} to obtain C_R.

It is obvious that the multiplication table of every band with R as minimal $\mathcal{D}$-class is a solution of B_R. Given a solution of B_R all elements of R in the corresponding band are $\mathcal{D}$-related due to constraint (8) and are in the minimal $\mathcal{D}$-class due to constraint (9). Elements in the complement of R are not $\mathcal{D}$-related to elements in R, as constraint (9) implies that their two-sided ideals differ. Consequently, the solutions of B_R are exactly the bands on $[n]$ having R as their minimal $\mathcal{D}$-class.

Let $\mathcal{R}_n^k$ denote the rectangular bands on all subsets of $[n]$ of size k and let $\mathcal{R}_n = \cup_{k=1}^{n} \mathcal{R}_n^k$. We then define the family of CSPs $\mathcal{L}_{\mathcal{R}_n} = \{B_R \mid R \in \mathcal{R}_n\}$ which fulfils the conditions of Lemma 1. It follows that each (anti-)isomorphism type of band will appear as a solution of exactly one of the CSPs in $\mathcal{L}_{\overline{\mathcal{R}}_n} = \{B_R \mid R \in \overline{\mathcal{R}}_n\}$ where $\overline{\mathcal{R}}_n$ denotes a set of representatives of rectangular bands of order at most n up to (anti-)isomorphism.

Lemma 2 allows us to compute the symmetries of a CSP B_R as a stabiliser of literals. We see that an element in $S_n \times C_2$ is a symmetry if and only if its restriction to R is an automorphism or anti-automorphism. Adding the canonicity constraint (5) for every non-identity element in the symmetry group then yields $\overline{B}_R$. The number of constraints added is maximal when R is the left (or right) zero semigroup on $[n]$, but then R is the unique solution of B_R because constraint (8) covers the whole multiplication table. As no actual search is needed in this case, replacing $\overline{L}_{\mathrm{id}_n}$ with the family of CSPs

$$\overline{\mathcal{L}}_{\overline{\mathcal{R}}_n} = \{\overline{B}_R \mid R \in \overline{\mathcal{R}}_n\} \tag{10}$$

strictly reduces the number of symmetries involved, thereby reducing the effect of the computational bottleneck discussed at the ends of Sections 3.2 and 4.

Table 1 Numbers of semigroups on $[n]$ up to (anti-)isomorphism

n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
#	4	18	126	1 160	15 973	836 021	1 843 120 128	52 989 400 714 478
e	*by number e of idempotents*							
1	2	5	19	132	3 107	623 615	1 834 861 133	52 976 551 026 562
2	2	7	37	216	1 780	32 652	4 665 709	12 710 266 442
3		6	44	351	3 093	33 445	600 027	68 769 167
4			26	326	4 157	53 145	754 315	14 050 493
5				135	2 961	56 020	1 007 475	18 660 074
6					875	30 395	822 176	20 044 250
7						6 749	348 692	12 889 961
8							60 601	4 389 418
9								618 111
d	*by minimal generator number d*							
1	2	3	4	5	6	7	8	9
2	2	11	48	149	441	1 230	3 464	9 945
3		4	65	588	4 506	27 743	156 898	911 672
4			9	397	8 370	549 037	18 014 631	240 061 550
5				21	2 600	239 410	1 774 277 445	791 830 876 983
6					50	18 474	50 525 311	52 140 869 887 616
7						120	142 082	56 457 790 001
8							289	1 176 005
9								697

6 The semigroups of order 9

We have used the families of CSPs introduced in the previous sections to obtain canonical representatives for semigroups of order 9 which are not nilpotent of degree 3 up to (anti-)isomorphism. More precisely, we solved the CSPs in

$$\overline{\mathcal{L}}_{\overline{\mathcal{F}}_9}^{-3} \setminus \left\{\overline{L}_{\mathrm{id}_9}\right\} \text{ and } \overline{\mathcal{L}}_{\overline{\mathcal{R}}_9}, \tag{11}$$

as defined in (7) and (10) obtaining a total of 23 161 651 504 solutions. The semigroups not searched for were the zero semigroup and the nilpotent semigroups of degree 3 of order 9. The number of the latter is 52 966 239 062 973 [7, Table 4]. All together there are 52 989 400 714 478 semigroups of order 9 up to (anti-)isomorphism of which almost 99.96% are nilpotent of degree 3.

To perform the computations we used GAP [11] and Minion [12]; the former to calculate stabilisers and also for the automated creation of the input files; the latter to solve the CSPs. The computations took around 87 hours on a machine with 2.66 GHz Intel X-5430 processor and 8 GB RAM. The code can be found in [4, Appendix C].

6.1 Classification

We analysed the semigroups obtained by search to extract various classification results. The numbers of semigroups with 9 elements sorted by their number

Table 2 Numbers of semigroups of order 9 with various properties

Idpts	self-dual	commutative	regular	inverse	comm.-inv.
1	613 365 656	9 940 825	2	2	2
2	8 265 721	664 080	23	23	16
3	739 317	249 330	148	129	111
4	410 158	222 637	830	567	504
5	328 937	201 060	4 136	1 750	1 555
6	223 226	148 647	17 535	3 870	3 460
7	113 160	82 481	66 822	6 582	6 137
8	38 979	30 789	217 437	7 505	7 505
9	7 510	5 994	618 111	5 994	5 994
$\sum$	623 492 664	11 545 843	925 044	26 422	25 284

of idempotents are listed in Table 1 together with numbers for lower orders from [21, Table 4.1], which we also verified using our search method. Also listed in the table are numbers of semigroups by their minimal generator number. For nilpotent semigroups of degree 3 these numbers are easily calculated from the summands of the formula given in [7, Theorem 2.3] using the fact that every nilpotent semigroup is generated by its indecomposable elements.

Information on the classification of semigroups of order 9 by certain properties is summarised in Table 2. The total number of commutative semigroups agrees with the result from [13]. The selection of properties was largely inspired by [21, Table 4.2] except that we also report numbers of self-dual semigroups. We determined the latter using the method described in the second paragraph of Section 3.3 to the CSPs from (11). In addition we needed the number of self-dual semigroups of order 9 that are nilpotent of degree 3 which is 606 097 491 [7, Table 5]. Note that except for the regular semigroups all classes listed in Table 2 consist entirely of self-dual semigroups.

6.2 Up to isomorphism

As mentioned in the introduction we also obtained results for the classification of semigroups up to isomorphism. In general this is achieved by replacing the group $S_n \times C_2$ wherever it appears in the considerations regarding symmetries of the CSPs with the group S_n. In many situations we can alternatively take advantage of the fact that we determined self-dual semigroups: twice the number of semigroups up to (anti-)isomorphism minus the number of self-dual semigroups yields the number of semigroups up to isomorphism. Hence there are 105 978 177 936 292 semigroups of order 9 up to isomorphism. These semigroups together with those of lower orders are classified by their number of idempotents and by their minimal generator number in Table 3.

Table 3 Numbers of semigroups on $[n]$ up to isomorphism

n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
#	5	24	188	1 915	28 634	1 627 672	3 684 030 417	105 978 177 936 292
e					*by number e of idempotents*			
1	2	5	20	171	5 284	1 224 331	3 667 785 000	105 952 488 687 468
2	3	9	50	309	2 806	58 583	9 207 430	25 412 267 163
3		10	72	590	5 422	61 323	1 150 085	136 799 017
4			46	594	7 772	101 539	1 466 691	27 690 828
5				251	5 668	109 107	1 983 558	36 991 211
6					1 682	59 576	1 626 956	39 865 274
7						13 213	690 871	25 666 762
8							119 826	8 739 857
9								1 228 712
d					*by minimal generator number d*			
1	2	3	4	5	6	7	8	9
2	3	14	64	212	664	1 930	5 678	17 010
3		7	103	954	7 835	50 541	294 622	1 751 293
4			17	703	15 144	1 075 353	35 850 090	479 050 352
5				41	4 886	463 784	3 546 839 307	1 583 613 947 364
6					99	35 818	100 760 203	104 281 178 828 643
7						239	279 932	112 902 004 698
8							577	2 335 530
9								1 393

7 Automorphism groups and distinct semigroups on a set

To determine the automorphism groups of semigroups with at most 9 elements, we use the idea described in the first paragraph of Section 3.3. Technically there are two different methods: we can record for each semigroup the isomorphisms that are automorphisms, or we can perform one search for every possible automorphism group. Neither approach is by itself feasible for $n = 9$, because of the large numbers of semigroups with 9 elements and of subgroups of S_9. We therefore took a mixed approach distinguishing the following mutually exclusive cases depending on the orders of automorphisms a semigroup S allows.

(i) $\mathrm{Aut}(S) \cong C_2^k$, $k \in \mathbb{N}$: Require strict inequality in all constraints that do not correspond to a permutation of order 2. Require also that equality holds for exactly $2^k - 1$ of the remaining constraints.
(ii) $|\mathrm{Aut}(S)| = 2^k$, $k \in \mathbb{N}$, but $\mathrm{Aut}(S) \not\cong C_2^k$: Require strict inequality in all constraints that do not correspond to a permutation of order $2^m, m \in \mathbb{N}$. Require also that equality holds for at least one constraint corresponding to a permutation of order 4.
(iii) $|\mathrm{Aut}(S)|$ contains an odd prime factor: Require that equality holds for at least one constraint corresponding to a permutation of odd prime order.

Semigroups covered by none of the three cases must have the trivial group as automorphism group. In the first case $\lfloor n/2 \rfloor$ is an upper bound for k because

Table 4 Automorphism groups of semigroups of order 2

automorphism group	ID	number up to (anti-)isomorphism	number up to isomorphism
trivial	$(1,1)$	3	3
C_2	$(2,1)$	1	2

Table 5 Automorphism groups of semigroups of order 3

automorphism group	ID	number up to (anti-)isomorphism	number up to isomorphism
trivial	$(1,1)$	12	15
C_2	$(2,1)$	5	7
S_3	$(6,1)$	1	2

Table 6 Automorphism groups of semigroups of order 4

automorphism group	ID	number up to (anti-)isomorphism	number up to isomorphism
trivial	$(1,1)$	78	112
C_2	$(2,1)$	39	62
$C_2 \times C_2$	$(4,2)$	3	5
S_3	$(6,1)$	5	7
S_4	$(24,12)$	1	2

Table 7 Automorphism groups of semigroups of order 5

automorphism group	ID	number up to (anti-)isomorphism	number up to isomorphism
trivial	$(1,1)$	746	1221
C_2	$(2,1)$	342	576
C_3	$(3,1)$	2	2
C_4	$(4,1)$	1	1
$C_2 \times C_2$	$(4,2)$	26	46
S_3	$(6,1)$	33	51
D_8	$(8,3)$	1	2
D_{12}	$(12,4)$	4	8
S_4	$(24,12)$	4	6
S_5	$(120,34)$	1	2

C_2^k is a subgroup of S_n if and only if $2k \leq n$ (see [14, Theorem 2]). We run a separate computation for each admissible value of k, specifying a unique isomorphism type of automorphism group, and record the number of solutions. In the other two cases we let Minion output a list of automorphisms for each solution and read it into GAP. We then use the identification function in the SmallGroups library [2] to find the isomorphism types of the groups. Note that it was not possible to exclude nilpotent semigroups of degree 3 from the various searches as their numbers with prescribed automorphism groups are unknown.

Table 8 Automorphism groups of semigroups of order 6

automorphism group	ID	number up to (anti-)isomorphism	number up to isomorphism
trivial	$(1, 1)$	10 965	19 684
C_2	$(2, 1)$	4 121	7 397
C_3	$(3, 1)$	26	32
C_4	$(4, 1)$	7	7
$C_2 \times C_2$	$(4, 2)$	441	806
S_3	$(6, 1)$	300	506
D_8	$(8, 3)$	17	30
$C_2 \times C_2 \times C_2$	$(8, 5)$	6	12
D_{12}	$(12, 4)$	49	92
S_4	$(24, 12)$	30	48
$S_3 \times S_3$	$(36, 10)$	2	4
$C_2 \times S_4$	$(48, 48)$	4	8
S_5	$(120, 34)$	4	6
S_6	$(720, 763)$	1	2

Table 9 Automorphism groups of semigroups of order 7

automorphism group	ID or order	number up to (anti-)isomorphism	number up to isomorphism
trivial	$(1, 1)$	746 277	1 458 882
C_2	$(2, 1)$	76 704	144 879
C_3	$(3, 1)$	412	620
C_4	$(4, 1)$	82	101
$C_2 \times C_2$	$(4, 2)$	7 314	13 756
C_5	$(5, 1)$	6	6
S_3	$(6, 1)$	3 638	6 552
C_6	$(6, 2)$	37	53
$C_4 \times C_2$	$(8, 2)$	4	6
D_8	$(8, 3)$	169	282
$C_2 \times C_2 \times C_2$	$(8, 5)$	172	330
D_{10}	$(10, 1)$	2	2
D_{12}	$(12, 4)$	790	1 476
$C_2 \times D_8$	$(16, 11)$	10	20
S_4	$(24, 12)$	277	475
$C_2 \times C_2 \times S_3$	$(24, 14)$	14	28
$S_3 \times S_3$	$(36, 10)$	24	44
$C_2 \times S_4$	$(48, 48)$	45	86
$(S_3 \times S_3) : C_2$	$(72, 40)$	1	2
S_5	$(120, 34)$	30	48
$S_3 \times S_4$	$(144, 183)$	4	8
$C_2 \times S_5$	$(240, 189)$	4	8
S_6	$(720, 763)$	4	6
S_7	5040	1	2

Table 10 Automorphism groups of semigroups of order 8

automorphism group	ID or order	number up to (anti-)isomorphism	number up to isomorphism
trivial	$(1,1)$	1 834 638 770	3 667 253 972
C_2	$(2,1)$	8 176 697	16 194 638
C_3	$(3,1)$	17 297	31 567
C_4	$(4,1)$	1 270	1 907
$C_2 \times C_2$	$(4,2)$	188 316	363 902
C_5	$(5,1)$	92	110
S_3	$(6,1)$	69 275	131 242
C_6	$(6,2)$	1 249	2 086
$C_4 \times C_2$	$(8,2)$	105	153
D_8	$(8,3)$	2 238	3 876
$C_2 \times C_2 \times C_2$	$(8,5)$	5 324	10 255
$C_3 \times C_3$	$(9,2)$	5	6
D_{10}	$(10,1)$	28	34
D_{12}	$(12,4)$	13 583	25 883
$C_2 \times D_8$	$(16,11)$	263	490
$C_2 \times C_2 \times C_2 \times C_2$	$(16,14)$	15	29
$C_3 \times S_3$	$(18,3)$	40	56
$C_5 : C_4$	$(20,3)$	1	1
$C_4 \times S_3$	$(24,5)$	4	6
S_4	$(24,12)$	3 461	6 293
$C_2 \times A_4$	$(24,13)$	4	4
$C_2 \times C_2 \times S_3$	$(24,14)$	491	966
$S_3 \times S_3$	$(36,10)$	368	674
$D_8 \times S_3$	$(48,38)$	11	22
$C_2 \times S_4$	$(48,48)$	768	1 445
$(S_3 \times S_3) : C_2$	$(72,40)$	16	28
$C_2 \times S_3 \times S_3$	$(72,46)$	12	24
$C_2 \times C_2 \times S_4$	$(96,226)$	14	28
S_5	$(120,34)$	277	475
$S_3 \times S_4$	$(144,183)$	44	84
$\mathrm{PSL}(3,2)$	$(168,42)$	1	1
$C_2 \times S_5$	$(240,189)$	44	84
$S_4 \times S_4$	$(576,8653)$	2	4
S_6	$(720,763)$	30	48
$S_5 \times S_3$	$(720,767)$	4	8
$C_2 \times S_6$	$(1440,5842)$	4	8
S_7	5040	4	6
S_8	40320	1	2

Tables 4, 5, 6, 7, 8, 9, 10, and 11 list the automorphism groups of semigroups up to (anti-)isomorphism and up to isomorphism with 2 to 9 elements. There is one table for each order, containing one line for each isomorphism type of automorphism group. The groups are identified by their ID in the SmallGroups library [2], if their order is less than 2000. In all cases a structural description, computed using the GAP command `StructureDescription`, is also given. Finally, the numbers of semigroups up to (anti-)isomorphism and up to isomorphism with the given group as automorphism group are provided. The numbers for semigroups up to (anti-)isomorphism of order at most 7 agree with those from [1], except for an obviously typographic omission of $C_2 \times C_2$ as

Table 11 Automorphism groups of semigroups of order 9

automorphism group	ID or order	number up to (anti-)isomorphism	number up to isomorphism
trivial	$(1,1)$	52 961 873 362 324	105 923 135 799 007
C_2	$(2,1)$	27 478 363 462	54 944 831 554
C_3	$(3,1)$	6 329 218	12 562 447
C_4	$(4,1)$	53 591	97 613
$C_2 \times C_2$	$(4,2)$	33 882 706	67 399 096
C_5	$(5,1)$	1 547	2 295
S_3	$(6,1)$	7 886 998	15 634 673
C_6	$(6,2)$	94 521	180 353
C_7	$(7,1)$	18	18
$C_4 \times C_2$	$(8,2)$	3 286	5 478
D_8	$(8,3)$	59 672	110 744
$C_2 \times C_2 \times C_2$	$(8,5)$	203 597	396 962
$C_3 \times C_3$	$(9,2)$	291	449
D_{10}	$(10,1)$	420	626
C_{10}	$(10,2)$	108	156
C_{12}	$(12,2)$	26	34
A_4	$(12,3)$	3	3
D_{12}	$(12,4)$	354 352	689 994
$C_6 \times C_2$	$(12,5)$	850	1 496
D_{14}	$(14,1)$	4	4
$C_4 \times C_2 \times C_2$	$(16,10)$	18	32
$C_2 \times D_8$	$(16,11)$	5 530	10 252
$C_2 \times C_2 \times C_2 \times C_2$	$(16,14)$	1 345	2 654
$C_3 \times S_3$	$(18,3)$	1 286	2 135
$(C_3 \times C_3) : C_2$	$(18,4)$	1	2
$C_5 : C_4$	$(20,3)$	8	9
D_{20}	$(20,4)$	36	52
$C_7 : C_3$	$(21,1)$	2	2
$C_4 \times S_3$	$(24,5)$	105	153
$C_3 \times D_8$	$(24,10)$	26	36
S_4	$(24,12)$	67 321	128 046
$C_2 \times A_4$	$(24,13)$	57	69
$C_2 \times C_2 \times S_3$	$(24,14)$	15 150	29 589
$C_4 \times D_8$	$(32,25)$	1	2
$(C_2 \times C_2 \times C_2 \times C_2) : C_2$	$(32,27)$	10	19
$C_2 \times C_2 \times D_8$	$(32,46)$	83	166
$S_3 \times S_3$	$(36,10)$	6 429	12 123
$\mathrm{GL}(2,3)$	$(48,29)$	1	1
$D_8 \times S_3$	$(48,38)$	263	486
$C_2 \times S_4$	$(48,48)$	13 204	25 243
$C_2 \times C_2 \times C_2 \times S_3$	$(48,51)$	44	88
$D_8 \times D_8$	$(64,226)$	1	1
$(S_3 \times S_3) : C_2$	$(72,40)$	158	263
$C_3 \times S_4$	$(72,42)$	34	50
$C_2 \times S_3 \times S_3$	$(72,46)$	474	940
$C_4 \times S_4$	$(96,186)$	4	6
$C_2 \times C_2 \times S_4$	$(96,226)$	479	946
S_5	$(120,34)$	3 454	6 281
$S_3 \times S_4$	$(144,183)$	705	1 327
$C_2 \times ((S_3 \times S_3) : C_2)$	$(144,186)$	11	22
$\mathrm{PSL}(3,2)$	$(168,42)$	3	3
$D_8 \times S_4$	$(192,1472)$	10	20
$S_3 \times S_3 \times S_3$	$(216,162)$	4	8
$C_2 \times S_5$	$(240,189)$	755	1 423
$C_2 \times S_3 \times S_4$	$(288,1028)$	24	48
$((((C_2 \times D_8) : C_2) : C_3) : C_2) : C_2$	$(384,5602)$	1	2
$C_2 \times C_2 \times S_5$	$(480,1186)$	14	28
$S_4 \times S_4$	$(576,8653)$	20	38
S_6	$(720,763)$	277	475
$S_5 \times S_3$	$(720,767)$	44	84
$(S_4 \times S_4) : C_2$	$(1152,157849)$	1	2
$C_2 \times S_6$	$(1440,5842)$	44	84
$S_5 \times S_4$	2880	4	8
$S_6 \times S_3$	4320	4	8
S_7	5040	30	48
$C_2 \times S_7$	10080	4	8
S_8	40320	4	6
S_9	362880	1	2

Table 12 Numbers of distinct semigroups on $[n]$

n	semigroups on $[n]$
2	8
3	113
4	3 492
5	183 732
6	17 061 118
7	7 743 056 064
8	148 195 347 518 186
9	38 447 365 355 811 944 462

automorphism group for semigroups of order 5. The numbers for semigroups up to (anti-)isomorphism of order 9 partially differ from those in [4, Table A.15], where some semigroups belonging to Case (iii) above were incorrectly counted as having trivial automorphism group.

The Minion computations to obtain the results took nearly two months on our machine with 2.66 GHz Intel X-5430 processor. To reduce the possibility of an error we confirmed the numbers in a second run using a different setup. Details about the code used to compute the automorphism groups can be found in [4, Appendix C.2.3].

An immediate observation is that most of the semigroups have trivial automorphism group and their ratio to all semigroups seems to converge to 1 with increasing order, thus supporting a conjecture from [7].

We further use our results to deduce the numbers of distinct semigroups on sets with 2 to 9 elements. For a semigroup S the number of isomorphic semigroups on the same underlying set equals $|S|!/|\operatorname{Aut}(S)|$. Hence the number of distinct semigroups on a set with n elements equals

$$n! \sum_{S} \frac{1}{|\operatorname{Aut}(S)|}$$

where the summation runs over a set of representatives of semigroups of order n up to isomorphism. New results in Table 12 are the numbers for orders 8 and 9, for lower orders we confirm the numbers available from [19, Sequence A023814].

While it is known that every group appears as the automorphism group of some semigroup, the number of isomorphism types of automorphism groups is small in comparison with the number of all isomorphism types of subgroups of the symmetric group (Table 13). Information about which types of automorphism groups appear could be useful in the development of algorithms to compute the automorphism group of a given semigroup. We observe in particular that for $2 \leq n \leq 9$ only rectangular bands have a transitive subgroup of S_n as automorphism group. We complete this section by showing that this statement holds for every order.

Proposition 1 *Let R be a finite semigroup. Then the following are equivalent:*

Table 13 Comparison of the numbers of isomorphism types of (a) subgroups of the symmetric group of degree n with (b) automorphism groups of semigroups of order n

n	1	2	3	4	5	6	7	8	9
(a) subgroups of S_n	1	2	4	9	16	29	55	137	241
(b) $\mathrm{Aut}(S)$ for $\vert S\vert = n$	1	2	3	5	10	14	24	38	65

(i) R is a rectangular band.
(ii) The automorphism group of R acts transitively on R.

Proof (i) $\Rightarrow$ (ii): Let $R = I \times \Lambda$. Then every element in the direct product $S_I \times S_\Lambda$ is an automorphism of R.

(ii) $\Rightarrow$ (i): Let $e \in R$ be an idempotent. For every $a \in R$ there exists an automorphism π of R that sends e to a. Hence R consists entirely of idempotents. As a band R is a semilattice of rectangular bands. Clearly, the set of rectangular bands, that is the set of $\mathcal{D}$-classes of R, is preserved by every automorphism. Hence every automorphism induces an automorphism of the semilattice. The automorphism group of a finite semilattice is transitive if and only if the semilattice is trivial. Therefore R consists of a single rectangular band. □

We conclude noting that the previous proposition implies that there are arbitrarily high orders, all prime numbers, for which the full symmetric group is the only transitive automorphism group for a semigroup of the given order.

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